# WHEN IS THE UNDERLYING SPACE OF AN ORBIFOLD A MANIFOLD WITH BOUNDARY?

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ABSTRACT. We answer the question of when the underlying space of an orbifold is a manifold with boundary in several categories.

## 1. INTRODUCTION

The question posed by Davis "When is the underlying space of a smooth orbifold a topological manifold?" [5, p. 9] amounts to the classification of finite subgroups of the orthogonal group $\mathrm{O}_n$ for which the quotient space $\mathbb{R}^n/G$ is a topological manifold and has been completely answered in [12], [11] and [10]. The quotient space $\mathbb{R}^n/G$ not only inherits a topology from $\mathbb{R}^n$ but also other structures, i.e. a metric and a piecewise linear structure. One may as well ask when $\mathbb{R}^n/G$ is a manifold with respect to such an additional structure. In case of the quotient metric this task translates to the question of when a Riemannian orbifold is a Lipschitz manifold. Moreover, it makes sense to admit manifolds with boundary in the formulation of Davis's question.

We call a finite group generated by *reflections* and *rotations* in a finite-dimensional Euclidean space, i.e. by orthogonal transformations with codimension *one* and *two* fixed point subspaces, a *reflection-rotation group*. We call it a rotation group if it is generated by rotations. A classification of reflection-rotation groups is contained in [12]. Based on earlier results (cf. [15], [11, 10]) we answer the question posed in the title in the following ways. Theorem A has already been obtained in [11], we state it for completeness.

**Theorem A (B).** For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a PL (Lipschitz) manifold with boundary if and only if $G$ is a reflection-rotation group. In this case $\mathbb{R}^n/G$ is either PL (bi-Lipschitz) homeomorphic to the half space $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ and $G$ contains a reflection or $\mathbb{R}^n/G$ is PL (bi-Lipschitz) homeomorphic to $\mathbb{R}^n$ and $G$ does not contain a reflection.

In particular, the underlying space of a Riemannian orbifold is a Lipschitz manifold if and only if all its local groups are reflection-rotation groups. The quotient space $S^3/P$ of a 3-sphere by the binary icosahedral group $P < \mathrm{SO}_4$ is Poincaré's homology sphere and it follows from Cannon's double suspension theorem that its double suspension $\Sigma^2(S^3/P)$ is a topological 5-sphere [4]. We refer to the realization of the binary icosahedral group in $\mathrm{SO}_4$ as *Poincaré group*. The answer in the topological category reads as follows.

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**Theorem C.** For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a topological manifold with boundary if and only if $G$ has the form

$$G = G_{rr} \times P_1 \times \ldots \times P_k$$

for a reflection-rotation group $G_{rr}$ and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces and such that $n > 4$ if $k = 1$ and $n > 5$ if $G$ contains in addition a reflection. In this case $\mathbb{R}^n/G$ is either homeomorphic to the half space $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ and $G$ contains a reflection or $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$ and $G$ does not contain a reflection.

The if directions of Theorem B and C follow from the if direction of Theorem A which has been proven in [11] (cf. [15]) and the double suspension theorem. To show the only-if direction we first prove

**Theorem D.** For a finite subgroup $G < \mathrm{O}_n$ the quotient space $\mathbb{R}^n/G$ is a homology manifold with boundary if and only if $G$ has the form

$$G = G_{rr} \times P_1 \times \ldots \times P_k$$

for a reflection-rotation group $G_{rr}$ and Poincaré groups $P_i < \mathrm{SO}_4$, $i = 1, \ldots, k$, such that the factors act in pairwise orthogonal spaces. In this case the boundary of $\mathbb{R}^n/G$ is nonempty if and only if $G_{rr}$ contains a reflection.

The following related question was asked by Vinberg.

**Question.** Let $G < \mathrm{O}_n$ be a rotation group. Is it always possible to find $n$ polynomials $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]^G$ in the invariant ring of $G$ for which the induced map

$$f = [f_1, \ldots, f_n] : \mathbb{R}^n/G \to \mathbb{R}^n$$

defines a homeomorphism?

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## 2. Preliminaries

2.1. **Homology manifolds.** Homology manifolds are generalizations of topological manifolds (cf. [3]). In order to simplify our proofs we make the following modified definition.

**Definition 2.1.** We say that a Hausdorff space $X$ is a *homology $n$-manifold*, if all its local homology groups coincide with the local homology groups of $\mathbb{R}^n$, i.e. if for all $x \in X$

$$H_i(X, X - \{x\}) = \begin{cases} 0, & \text{for } i \neq n \\ \mathbb{Z}, & \text{for } i = n \end{cases}$$

holds.

**Remark 2.2.** All spaces occurring in this paper are finite-dimensional simplicial complexes. Every such space is a finite-dimensional absolute neighborhood retract [13, Application 18.4, p. 61]. Therefore, our modification of other stricter definitions of homology manifolds (cf. [3], [20]) does not make any difference for the formulation of Theorem D.

For a topological space $X$ and a subspace $Y \subset X$ we define the *double of $X$ along $Y$* to be

$$2_Y X = X \times \{0,1\}/\sim \text{ where } (y,0) \sim (y,1) \text{ for all } y \in Y$$

endowed with the quotient topology and we simply denote it by $2X$ if the meaning of the subspace is clear. In order to deal with Davis's question for manifolds with boundary, we define homology manifolds with boundary in the following way.

**Definition 2.3.** We say that a Hausdorff space $X$ is a *homology $(n+1)$-manifold with boundary*, if it can be decomposed into a nonempty set of interior points $\mathring{X}$ and a set of boundary points $\partial X$ such that its double $2X$ along its boundary is a homology $(n+1)$-manifold, its boundary $\partial X$ is either empty or a homology $n$-manifold and the local homology groups at boundary points coincide with those of $0 \in \partial(\mathbb{R}^n \times \mathbb{R}_{\leq 0})$, i.e. for all $x \in \partial X$ and all $i \geq 0$ we have $H_i(X, X - \{x\}) = 0$.

**Remark 2.4.** If the space $X$ in the definition of a homology manifold with boundary is sufficiently nice, then its boundary and its double are automatically homology manifolds. This is for example the case if $X$ is a PL space (cf. [14, p. 510], [16, Prop. 5.4.11, p. 188]) and so it holds for all spaces we are working with in this paper (cf. [11]).

The open cone of a topological space $X$ is defined to be $CX = (X \times [0,1))/(X \times \{0\})$. Homology manifolds share the following properties. Proofs are standard computations (cf. [6, Cor. VI.12.10, p. 181], [1, Thm. 16.11, p. 378], [7, p. 117]).

**Lemma 2.5.** *For Hausdorff spaces $X$ and $Y$ the following statements hold for integers $n \geq 0$.*

- *(i) $X$ and $Y$ are homology manifolds, if and only if $X \times Y$ is a homology manifold.*
- *(ii) If $X$ is a homology manifold, then $X \times Y$ is a homology manifold with boundary, if and only if $Y$ is a homology manifold with boundary. Moreover, we have $\partial(X \times Y) = X \times \partial Y$.*
- *(iii) $CX$ is a homology $(n+1)$-manifold if and only if $X$ is a homology $n$-manifold and $H_*(X) = H_*(S^n)$.*
- *(iv) $CX$ is a homology $(n+2)$-manifold with nonempty boundary $C(\partial X)$ if and only if $X$ is a homology $(n+1)$-manifold with nonempty boundary and $H_*(X) = H_*(\{*\})$, $H_*(\partial X) = H_*(S^n)$.*

2.2. **Piecewise linear manifolds.** For a discussion of piecewise linear manifolds, we refer to the preliminary section of [11].

2.3. **Lipschitz manifolds.** We say that a metric space $X$ is a *Lipschitz manifold* if it is a topological manifold and the coordinate maps can be chosen to be bi-Lipschitz. For a finite subgroup $G < \mathrm{O}_n$ the quotient $\mathbb{R}^n/G$ inherits a metric from $\mathbb{R}^n$, the so-called *quotient metric*, where the distance between two points in $\mathbb{R}^n/G$ is defined to be the distance of the corresponding orbits in $\mathbb{R}^n$. With respect to this metric $\mathbb{R}^n/G$ is a *length space*, i.e. the distance between two points is the infimum of the lengths of all rectifiable paths connecting these points (cf. [2]). We would like to know when it is a Lipschitz manifold. Since $\mathbb{R}^n/G$ can be triangulated by a simplicial complex $K/G$ (cf. [11]), the metric on $\mathbb{R}^n/G$ can be recovered from the flat metrics on the simplices of $K/G$. Therefore, $\mathbb{R}^n/G$ is a Lipschitz manifold with boundary if it is a PL manifold with boundary. In particular, $\mathbb{R}^n/G$ is a Lipschitz manifold

with boundary if $G$ is a reflection-rotation group by Theorem A which has been proven in [11].

Siebenmann and Sullivan established the following necessary and sufficient condition for a simplicial complex to be a Lipschitz manifold [18, Thm. 1, p. 504; Thm. 2, p. 506 in combination with Remark ($i$), p. 507)].

**Theorem 2.6.** *A locally finite simplicial complex $K$ with a length metric induced by flat metrics on its simplices is a Lipschitz manifold, if and only if the link of every simplex of $K$ is a homotopy sphere and a Lipschitz manifold with respect to its induced length metric.*

According to this result, the same argument as in the PL category shows that $G$ is a rotation group, if $\mathbb{R}^n/G$ is a Lipschitz manifold (cf. [11]). First, the fact that the link of the origin in the triangulation $K$ of $\mathbb{R}^n$ is again a Lipschitz manifold, implies by induction that all proper isotropy groups in $G$ are rotation groups. Then the simply connectedness of this link implies that $G$ is generated by its isotropy groups and thus is a rotation group itself. Necessary conditions for the case that $\mathbb{R}^n/G$ is a Lipschitz manifold with nonempty boundary will be deduced in the next section.

## 3. Proof of the main results

For manifolds without boundary Theorem D has been proven in [10]. To prove it also in the general case for manifolds with boundary we first show a lemma based on the following concepts.

**Definition 3.1.** A metric space $\mathcal{O}$ is called a *Riemannian orbifold* of dimension $n$, if for each point $x \in \mathcal{O}$ there exists an open neighborhood $U$ of $x$ in $\mathcal{O}$ and an $n$-dimensional Riemannian manifold $M$ together with a finite group $G$ acting isometrically on $M$ such that $U$ and $M/G$ are isometric.

For a set of data $(x, U, M, G)$ as in the definition denote by $\pi_x : M \to U$ the composition of the natural projection from $M$ to $M/G$ and an isometry from $M/G$ to $U$ and let $p \in \pi_x^{-1}(x)$. Every Riemannian orbifold in the above sense is a smooth orbifold in the sense of [5, 19] with the $(B_r(0) \subset T_pM, G_p, B_r(x), \pi_x \circ \exp_p)$ for sufficiently small $r$ as orbifold charts.

**Definition 3.2.** A map $\varphi : \mathcal{O} \to \mathcal{O}'$ between Riemannian orbifolds is called an orbifold covering if any point $x \in \mathcal{O}'$ has a neighborhood $U$ isometric to some $M/G$ (cf. Definition 3.1) such that each component $U_i$ of $\varphi^{-1}(U)$ contains precisely one preimage of $x$ and is isometric to $M/G_i$ for some subgroup $G_i < G$.

Now we can show

**Lemma 3.3.** *Let $G < \mathrm{O}_n$ be a finite subgroup with orientation preserving subgroup $G^+$ and assume that $\mathbb{R}^n/G$ is a homology manifold with nonempty boundary. Then $G$ contains a reflection and there exists an isometry $\varphi$ from the double $2(\mathbb{R}^n/G)$ with its induced length metric to $\mathbb{R}^n/G^+$ such that $p_0 = p_1 \circ \varphi$ where $p_0$ and $p_1$ are the natural projections from $2(\mathbb{R}^n/G)$ and $\mathbb{R}^n/G^+$ to $\mathbb{R}^n/G$.*

*Proof.* The proof is by induction on $n$. For $n = 1, 2$ the claim is clear. Assume it holds for some fixed $n > 1$ and let $G < \mathrm{O}_{n+1}$ be a finite subgroup such that $\mathbb{R}^{n+1}/G$ is a homology manifold with nonempty boundary. Then $S^n/G$ is also a homology manifold with nonempty

boundary by Lemma 2.5. For a point $x \in S^n$ whose coset lies in the boundary of $S^n/G$ the quotient space $T_xS^n/G_x$ is a homology manifold with nonempty boundary. Therefore, it follows by induction that $G_x \subset G$ contains a reflection and that there exists an isometry $\tilde{\theta} : 2(T_xS^n/G_x) \to T_xS^n/G_x^+$ with the property stated in the lemma. Using the exponential map, we obtain an equivariant bijection $\theta : B_r(x_0) \to B_r(x_1)$ between small balls $B_r(x_0)$ and $B_r(x_1)$ about the cosets $x_0$ and $x_1$ of $x$ in $2(S^n/G_x)$ and $S^n/G_x^+$, respectively. By construction the map $\theta$ descends to an isometry, in fact to the identity, between the quotients of $B_r(x_0)$ and $B_r(x_1)$ by the respective reflection. Since the metrics on $S^n/G_x$, $2(S^n/G_x)$ and $S^n/G_x^+$ are length metrics this implies that the restriction $\theta : B_{r/4}(x_0) \to B_{r/4}(x_1)$ is an isometry. In particular, we see that $2(S^n/G)$ is a Riemannian orbifold and that the natural projection $p_0 : 2(S^n/G) \to S^n/G$ is a covering of Riemannian orbifolds. By the assumption $n > 1$ the sphere $S^n$ is simply connected. Therefore there exists an index 2 subgroup $\tilde{G} < G$ and a homeomorphism $\varphi : 2(S^n/G) \to S^n/\tilde{G}$ with $p_0 = p_1 \circ \varphi$ where $p_1 : S^n/\tilde{G} \to S^n/G$ is the natural projection [19, Chapt. 13, p. 305]. Similar as above we see that $\varphi$ is an isometry. Moreover, since $2(S^n/G) = S^n/\tilde{G}$ has the integral homology of a sphere (cf. Definition 2.3 and Lemma 2.5), the subgroup $\tilde{G}$ preserves the orientation [10, Lem. 2.14]. The fact that both $G^+$ and $\tilde{G}$ are orientation preserving subgroups of index 2 in $G$ implies $G^+ = \tilde{G}$. The linear extension $\varphi : 2(\mathbb{R}^{n+1}/G) \to \mathbb{R}^{n+1}/G^+$ of $\varphi : 2(S^n/G) \to S^n/G^+$ is an isometry that satisfies the desired property (note that $C(2(S^n/G)) = 2(C(S^n/G)) = 2((CS^n)/G)$ as metric spaces, cf. [2, Sect. 3.6.2.]). □

Now we are in the position to finish the proof of Theorem D.

*Proof of Theorem D.* The if direction follows from Lemma 2.5, (*ii*), and the result in [11] (cf. [10]). Conversely, assume that $G < \mathrm{O}_n$ is a finite subgroup such that $\mathbb{R}^n/G$ is a homology manifold with boundary. According to Lemma 3.3, our Definition 2.3 (cf. the subsequent remark) and the result in [10], the orientation preserving subgroup $G^+$ of $G$ is a product of a rotation group and a certain number of Poincaré groups. So we are done, if $G$ itself preserves the orientation. Otherwise $G$ contains a reflection $s$ by Lemma 3.3 which normalizes $G^+$. This reflection can only act in one of the factors of $G^+$. Therefore the claim follows, if we can show that $\mathbb{R}^4/\tilde{P}$ is not a homology manifold for $\tilde{P} = \langle P, s \rangle$ where $s$ is one of the existing reflections in the normalizer of $P$ in $\mathrm{O}_4$. The coset of $s$ in $\tilde{P}/P$ acts as an orientation reversing isometry on $S^3/P$. Hence, its fixed point subspace is a disjoint union of points and totally geodesic embedded surfaces. If $\mathbb{R}^4/\tilde{P}$ were a homology manifold, then only a single embedded sphere could occur and $S^3/P$ would be the double of $S^3/\tilde{P}$ along this sphere by Lemma 3.3. In this case $P$ would be a free product of isomorphic groups due to the theorem of Seifert and van Kampen on fundamental groups [7, Thm. 1.20., p. 43] (the boundary of $S^3/\tilde{P}$ admits a collar). This is a contradiction, since $P$ is neither trivial nor infinite and thus the claim follows. □

Now Theorem C can be proven as well.

*Proof of Theorem C.* According to the results in [10] and [11] the quotient space $\mathbb{R}^n/G$ is a topological manifold with boundary for all groups described in Theorem C. Conversely, suppose $G < \mathrm{O}_n$ is a finite subgroup for which $\mathbb{R}^n/G$ is a topological manifold with boundary. Then $G$ has the form $G = G_{rr} \times P_1 \times \ldots \times P_k$ as in Theorem *D* by that theorem with $n > 4$

for $k = 1$ (cf. [10]). Moreover, the additional condition $n > 5$ for $k = 1$ in the case that $G$ contains a reflection also holds, since $\mathbb{R}^4/P$, which would have to be the boundary of $\mathbb{R}_{\geq 0} \times \mathbb{R}^4/P$ by homological reasons (cf. Lemma 2.5), is not a topological manifold. $\square$

We conclude with a proof of Theorem B.

*Proof of Theorem B.* The if direction is a direct consequence of Theorem A (cf. Section 2.3). The only-if direction in the case in which the boundary is empty has been proven in Section 2.3. So assume that $G < \mathrm{O}_n$ is a finite subgroup for which $\mathbb{R}^n/G$ is a Lipschitz manifold with nonempty boundary. In view of Lemma 3.3 it suffices to observe that the double of a length space that is a Lipschitz manifold with boundary with its induced length metric is a Lipschitz manifold without boundary. $\square$

## References


[1] Glen E. Bredon, *Sheaf theory*, 2nd ed., Graduate Texts in Mathematics, vol. 170, Springer-Verlag, New York, 1997.

[2] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.

[3] J. W. Cannon, *The recognition problem: what is a topological manifold?*, Bull. Amer. Math. Soc. **84** (1978), no. 5, 832-866.

[4] ———, *Shrinking cell-like decompositions of manifolds*. Codimension three, Ann. of Math. (2) **110** (1979), no. 1, 83-112.

[5] Michael W. Davis, *Lectures on orbifolds and reflection groups*, Transformation groups and moduli spaces of curves, Adv. Lect. Math. (ALM), vol. 16, Int. Press, Somerville, MA, 2011, pp. 63-93.

[6] Albrecht Dold, *Lectures on algebraic topology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 200, Springer-Verlag, Berlin, 1980.

[7] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[8] J. F. P. Hudson, *Piecewise linear topology*, University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[9] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[10] Christian Lange, *When is the underlying space of an orbifold a topological manifold?*, preprint, arXiv:1307.4875v1 (2013).

[11] ———, *Characterization of finite groups generated by reflections and rotations*, preprint, arXiv (2015).

[12] Christian Lange and Marina A. Mikhaîlova, *Classification of finite groups generated by reflections and rotations*, preprint, arXiv (2015). To appear in Transformation Groups.

[13] Solomon Lefschetz, Topics in Topology, *Topics in Topology*, no. 10, Princeton University Press, Princeton, N. J., 1942.

[14] C. R. F. Maunder, *Algebraic topology*, Dover Publications, Inc., Mineola, NY, 1996. Reprint of the 1980 edition.

[15] M. A. Mikhaîlova, *On the quotient space modulo the action of a finite group generated by pseudoreflections*, Math. USSR Izvestiya **24** (1985), no. 1, 99–119. [A translation of Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), no. 1, 104–126 (Russian)].

[16] W. J. R. Mitchell, *Defining the boundary of a homology manifold*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 509–513,

[17] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69.

[18] L. Siebenmann and D. Sullivan, *On complexes that are Lipschitz manifolds*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York-London, 1979, pp. 503–525.

[19] William P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University Press, 1979.

[20] Shmuel Weinberger, *Homology manifolds*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 1085–1102.

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